

ON A FAMILY OF q -BINOMIAL DISTRIBUTIONS

MARTIN ZEINER

ABSTRACT. We introduce a family of q -analogues of the binomial distribution, which generalises the Stieltjes-Wigert-, Rogers-Szegö-, and Kemp-distribution. Basic properties of this family are provided and several convergence results involving the classical binomial, Poisson, discrete normal distribution, and a family of q -analogues of the Poisson distribution are established. These results generalise convergence properties of Kemp's-distribution, and some of them are q -analogues of classical convergence properties.

1. INTRODUCTION

In [8] Kemp studied many q -analogues of the classical binomial distribution, in particular she investigated Kemp's distribution, the Rogers-Szegö and the Stieltjes-Wigert distribution, which all are of the form

$$\mathbb{P}(X = x) = C_\alpha \cdot \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^x \quad x = 0 \dots n, 0 < \theta,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q, q)_n}{(q, q)_k (q, q)_{n-k}} \quad \text{and} \quad (z, q)_n = \prod_{i=0}^{n-1} (1 - zq^i)$$

are the q -binomial coefficient and the q -shifted factorial, and where C_α is a normalising constant. In this paper we are interested in the convergence properties of this family of q -binomial distributions. We will see that the behaviour in the case $\alpha = 0$ is very different from the case $\alpha > 0$. For Kemp's distribution (i.e. $\alpha = \frac{1}{2}$) the limit distributions are the Heine distribution and the discrete normal distribution. This was done by Gerhold and Zeiner [6]. We will show that these results can be generalized to the case $\alpha > 0$.

This paper is organised as follows: In Section 2 we give the definitions of the q -binomial distributions mentioned above and sum up their basic convergence properties. Afterwards we introduce the family \mathcal{B} of q -binomial distributions we are interested in and a family of q -Poisson distributions. Afterwards we study basic properties of the family \mathcal{B} in Section 3. In Sections 4-5 we investigate sequences of random variables X_n with $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$. In particular we show that there are analogues to the convergence of the classical binomial distribution to the Poisson distribution and the normal distribution, and that the limits $q \rightarrow 1$ and $n \rightarrow \infty$ can be exchanged. Section 4 deals with convergent parameter sequences, in particular with the case of constant parameter and constant mean, and contains a detailed analysis of the behaviour of the RS-distribution in the limit $\theta_n \rightarrow 1$. In Section 5 we examine the case of an increasing parameter sequence θ_n . We show that, if $\alpha > 0$ and θ_n grows not too fast, the normalised X_n converge to a discrete normal distribution.

2. PRELIMINARIES

Throughout this paper we use the notation of [5]. Kemp's distribution $KB(n, \theta, q)$ was introduced in [8] and is defined as

$$\mathbb{P}(X_{KB} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{x(x-1)/2}}{(-\theta, q)_n}, \quad 0 \leq x \leq n, 0 < \theta.$$

For properties and applications of this distribution see [6, 7, 10, 12]. In the limit $q \rightarrow 1$ Kemp's distribution converges to a binomial distribution:

$$KB(n, \theta, q) \rightarrow B\left(n, \frac{\theta}{1 + \theta}\right).$$

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If n goes to ∞ , Kemp's distribution tends to the Heine distribution $H(\theta)$, whose probabilities are given by

$$\mathbb{P}(X_H = x) = \frac{q^{x(x-1)/2}\theta^x}{(q, q)_x} e_q(-\theta), \quad x \geq 0,$$

where

$$e_q(z) = \frac{1}{(z, q)_\infty}, \quad z \in \mathbb{C} \setminus \{q^{-i} : i = 0, 1, 2, \dots\},$$

is a q -analogue of the exponential function, since $e_q((1-q)z) \rightarrow e^z$, as $q \rightarrow 1$. The Heine distribution converges to the Poisson-distribution in the sense that $H((1-q)\theta) \rightarrow P(\theta)$ for $q \rightarrow 1$ and can therefore be seen as a q -analogue of the Poisson distribution.

Kemp [10] also introduced two other q -analogues of the binomial distribution, namely the Rogers-Szegő- (RS) and the Stieltjes-Wigert-distribution (SW), whose probabilities are very similar to those of Kemp's-distribution:

$$\begin{aligned} \mathbb{P}(X_{RS} = x) &= C_{RS} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x, \quad 0 \leq x \leq n, \quad 0 < \theta, \\ \mathbb{P}(X_{SW} = x) &= C_{SW} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{x(x-1)} \theta^x, \quad 0 \leq x \leq n, \quad 0 < \theta, \end{aligned}$$

where C_{RS} and C_{SW} are normalising constants. For $q \rightarrow 1$ these distributions tend to a binomial distribution with parameter $\frac{\theta}{1+\theta}$. In the limit $n \rightarrow \infty$ the RS-distribution converges for $\theta < 1$ to an Euler distribution with parameter θ , which is given by

$$\mathbb{P}(X_E = x) = \frac{\theta^x}{(q, q)_x} E_q(-\theta), \quad x \geq 0,$$

where

$$E_q(z) = (-z, q)_\infty, \quad z \in \mathbb{C},$$

is an other q -analogue of the exponential function, since $E_q((1-q)z) \rightarrow e^z$, as $q \rightarrow 1$. Moreover, we have $e_q(z)E_q(-z) = 1$. The Euler distribution is a q -analogue of the Poisson distribution since $E((1-q)\theta) \rightarrow P(\theta)$.

Because of the similarities of these distributions we introduce a family \mathcal{B} of q -analogues of binomial distributions which covers the distributions mentioned above as special cases:

Definition 2.1. We say a random variable X is $\mathcal{B}(\alpha, \theta, n, q)$ -distributed iff

$$\mathbb{P}(X = x) = \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^x}{\sum_{y=0}^n \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2} \theta^y}, \quad x = 0, \dots, n, \quad 0 < \theta, \quad 0 \leq \alpha.$$

For $\alpha = 0$ this is the RS-distribution, $\alpha = \frac{1}{2}$ gives a $KB(n, \theta q^{1/2}, q)$ -distribution and $\alpha = 1$ a $SW(n, \theta q, q)$ -distribution.

Moreover, we define a family \mathcal{P} of q -analogues of the Poisson distribution by

$$\mathbb{P}(X = x) = \frac{q^{\alpha x^2} \theta^x}{(q, q)_x} \frac{1}{E_q^{2\alpha}(\theta)}, \quad 0 \leq x,$$

where $0 < \theta < 1$ if $\alpha = 0$, and $0 < \theta$ if $\alpha > 0$, and E_q^α is a q -analogue of the exponential function (which was introduced by [4] and studied by [2] and also appears in [3]) defined by

$$(1) \quad E_q^\alpha(z) = \sum_{x \geq 0} \frac{q^{\frac{\alpha}{2} x^2}}{(q, q)_x} z^x,$$

since in the limit $q \rightarrow 1$ we have $E_q^\alpha((1-q)z) \rightarrow e^z$. We then write $X \sim \mathcal{P}(\alpha, \theta, q)$. For $\alpha = 0$ we obtain the Euler distribution, and $\alpha = \frac{1}{2}$ gives a $H(\theta q^{1/2})$ -distribution. The sum in (1) has a different behaviour for $\alpha = 0$ and $\alpha > 0$: In the case $\alpha = 0$ it is convergent only for $0 \leq |z| < 1$, but for $\alpha > 0$ it converges for all $z \in \mathbb{C}$. This is why we restricted the parameter θ in the definition of our q -Poisson family. Consequently there is a big difference in the behaviour of the Rogers-Szegő-distribution and the other members of this q -binomial-family. So we will often distinguish between $\alpha = 0$ and $\alpha > 0$ in the convergence results.

3. PROPERTIES OF THE FAMILY \mathcal{B}

As noted above we study basic properties of our family \mathcal{B} . We show that it is in fact a q -analogue of the binomial distribution and logconcave. Analogous properties with respect to the Poisson distribution hold for the family \mathcal{P} too. Then we give a characterisation of a $\mathcal{B}(\alpha, \theta, n, q)$ -distribution and a random walk model for \mathcal{B} and then we turn to the study of the behaviour of the mean of a $\mathcal{B}(\alpha, \theta, n, q)$ -distribution in dependence on n , θ and α . In the present section we always allow $\alpha \geq 0$.

The following two theorems show that our families \mathcal{B} and \mathcal{P} tend to the classical binomial and Poisson distribution. This generalises the results for the Kemp-, SW-, RS-, Heine, and Euler distributions.

Theorem 3.1. *For $q \rightarrow 1$ we have*

$$\mathcal{B}(\alpha, \theta, n, q) \rightarrow B\left(n, \frac{\theta}{1+\theta}\right).$$

Proof. By definition,

$$\begin{aligned} \mathbb{P}(X = x) &= \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^x}{\sum_{y=0}^n \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2} \theta^y} \\ &\rightarrow \frac{\binom{n}{x} \theta^x}{\sum_{y=0}^n \binom{n}{y} \theta^y} = \frac{\binom{n}{x} \theta^x}{(1+\theta)^n} \\ &= \binom{n}{x} \left(\frac{\theta}{1+\theta}\right)^x \left(\frac{1}{1+\theta}\right)^{n-x}. \end{aligned} \quad \square$$

Theorem 3.2. *In the limit $q \rightarrow 1$ we have $\mathcal{P}(\alpha, (1-q)\theta, q) \rightarrow P(\theta)$.*

Proof. By definition,

$$\mathbb{P}(X = x) = \frac{q^{\alpha x^2} (1-q)^x \theta^x}{(q, q)_x} \frac{1}{E_q^{2\alpha}((1-q)\theta)} \rightarrow \frac{\theta^x}{x!} \exp(-\theta). \quad \square$$

Kemp showed in [10] that the RS-, SW-, and Kemp-distribution are logconcave, i.e.,

$$\Delta(x) := \frac{\mathbb{P}(X = x+1)}{\mathbb{P}(X = x)} - \frac{\mathbb{P}(X = x+2)}{\mathbb{P}(X = x+1)} > 0$$

for $x = 0, \dots, n-2$. We can generalise this as follows:

Theorem 3.3. *$\mathcal{B}(\alpha, \theta, n, q)$ is logconcave.*

Proof. We have

$$\begin{aligned} \Delta(x) &= \frac{q^{\alpha(x+1)^2} \theta^{x+1} (q, q)_x (q, q)_{n-x}}{(q, q)_{x+1} (q, q)_{n-x-1} q^{\alpha x^2} \theta^x} - \frac{q^{\alpha(x+2)^2} \theta^{x+2} (q, q)_{x+1} (q, q)_{n-x-1}}{(q, q)_{x+2} (q, q)_{n-x-2} q^{\alpha(x+1)^2} \theta^{x+1}} \\ &= \theta \left(\frac{q^{2\alpha x + \alpha} (1 - q^{n-x})}{1 - q^{x+1}} - \frac{(1 - q^{n-x-1}) q^{2\alpha x + 3\alpha}}{1 - q^{x+2}} \right) \\ &= \theta q^{2\alpha x + \alpha} \left(\frac{1 - q^{x+2} - q^{n-x} + q^{n+2} - (1 - q^{n-x-1} - q^{x+1} + q^n) q^{2\alpha}}{(1 - q^{x+1})(1 - q^{x+2})} \right). \end{aligned}$$

For $\alpha = 0$ we have $\Delta(x) > 0$ by [10], and the numerator is increasing in α since

$$1 - q^{n-x-1} - q^{x+1} + q^n = q^{n-x-1} (q^{x+1} - 1) - (q^{x+1} - 1) > 0$$

for $x < n-1$. □

In the same way we obtain the same property of the family \mathcal{P} .

Theorem 3.4. *$\mathcal{P}(\alpha, \theta, q)$ is logconcave.*

For the Heine- and the Euler-distribution this property was proven by Kemp [9].

In [11] Kemp characterised some q -analogues of the binomial distribution as the conditional distribution of $U|(U+V=m)$ where U and V are independent random variables. We can characterise our family \mathcal{B} in an analogous way and generalise some of Kemp's results.

Theorem 3.5. *A $\mathcal{B}(\alpha, \theta/\lambda, m, q)$ -distribution is the distribution of $U|(U+V=m)$, where U and V are independent, iff U has a $\mathcal{P}(\alpha, \beta, \theta)$ -distribution and V has an Euler-distribution with parameter λ .*

Proof. The proof runs along the same lines as the proofs in [11]: If U and V have the postulated distributions, then

$$\begin{aligned}\mathbb{P}(U = u|U + V = m) &= C \frac{\theta^u q^{\alpha u^2}}{(q, q)_u} \frac{\lambda^{m-u}}{(q, q)_{m-u}} \\ &= C \frac{\lambda^m}{(q, q)_u (q, q)_{m-u}} \left(\frac{\theta}{\lambda}\right)^u q^{\alpha u^2}.\end{aligned}$$

To prove the other implication, we need the following theorem ([13]):

Let X and Y be independent discrete random variables and

$$c(x, x + y) = \mathbb{P}(X = x|X + Y = x + y).$$

If

$$\frac{c(x + y, x + y)c(0, y)}{c(x, x + y)c(y, y)} = \frac{h(x + y)}{h(x)h(y)},$$

where h is a nonnegative function, then

$$f(x) = f(0)h(x)e^{ax}, \quad g(y) = g(0)k(y)e^{ay},$$

where a is an arbitrary parameter and

$$0 < f(x) = \mathbb{P}(X = x), \quad 0 < g(y) = \mathbb{P}(Y = y), \quad k(y) = \frac{h(y)c(0, y)}{c(y, y)}.$$

Here we have

$$\frac{c(u + v, u + v)c(0, v)}{c(u, u + v)c(v, v)} = \frac{\left(\frac{\theta}{\lambda}\right)^{u+v} q^{\alpha(u+v)^2}}{\frac{(q, q)_{u+v}}{(q, q)_v (q, q)_u} \left(\frac{\theta}{\lambda}\right)^u q^{\alpha u^2} \left(\frac{\theta}{\lambda}\right)^v q^{\alpha v^2}} = \frac{h(u + v)}{h(u)h(v)},$$

where

$$h(u) = \frac{q^{\alpha u^2}}{(q, q)_u}.$$

Thus $k(v) = (\theta/\lambda)^v/(q, q)_v$ and

$$\begin{aligned}\mathbb{P}(U = u) &= C_1 \frac{q^{\alpha u^2} e^{au}}{(q, q)_u}, \\ \mathbb{P}(V = v) &= C_2 \left(\frac{\theta e^a}{\lambda}\right)^v \frac{1}{(q, q)_v},\end{aligned}$$

yielding a $\mathcal{P}(\alpha, e^a, q)$ -distribution and an Euler distribution. \square

We now give a random-walk-model for the family \mathcal{B} (the models for the Kemp-, RS-, and SW-distribution given in [10] are special cases of this model). Let a_x and b_x denote the probabilities to move up resp. down and choose

$$a_x = c\gamma q^{2\alpha x} (1 - q^{n-x}) \quad \text{and} \quad b_x = c(1 - q^x)$$

for $x = 0, \dots, n$. Then $\mathcal{B}(\alpha, \gamma q^{-\alpha}, n, q)$ is a stationary distribution. To see this, note that for a stationary distribution we must have

$$\mathbb{P}(X = x) = \mathbb{P}(X = x)(1 - a_x - b_x) + \mathbb{P}(X = x + 1)b_{x+1} + \mathbb{P}(X = x - 1)a_{x-1}.$$

So we have to show that

$$\delta(x) := -\mathbb{P}(X = x)(a_x + b_x) + \mathbb{P}(X = x + 1)b_{x+1} + \mathbb{P}(X = x - 1)a_{x-1} = 0$$

if $X \sim \mathcal{B}(\alpha, \gamma q^{-\alpha}, n, q)$. For $1 \leq x \leq n - 1$ we have

$$\begin{aligned}\delta(x) &= C \left(- \left[\begin{matrix} n \\ x \end{matrix} \right]_q q^{\alpha x^2} \gamma^x q^{-\alpha x} (c(1 - q^x) + c\gamma q^{2\alpha x} (1 - q^{n-x})) \right. \\ &\quad + \left[\begin{matrix} n \\ x - 1 \end{matrix} \right]_q q^{\alpha(x-1)^2} \gamma^{x-1} q^{-\alpha(x-1)} c\gamma q^{2\alpha(x-1)} (1 - q^{n-x+1}) \\ &\quad \left. + \left[\begin{matrix} n \\ x + 1 \end{matrix} \right]_q q^{\alpha(x+1)^2} \gamma^{x+1} q^{-\alpha(x+1)} c(1 - q^{x+1}) \right).\end{aligned}$$

Using the relation

$$\left[\begin{matrix} n \\ x - 1 \end{matrix} \right]_q (1 - q^{n-x+1}) = \left[\begin{matrix} n \\ x \end{matrix} \right]_q (1 - q^x)$$

we obtain that the terms with γ^x and γ^{x+1} vanish. Similarly $\delta(0)$ and $\delta(n)$ can be treated.

Now we study the means; for this purpose let us denote by $\mu_n(\alpha, \theta, q)$ the mean of a random variable $X \sim \mathcal{B}(\alpha, \theta, n, q)$. The following lemmas are devoted to the behaviour of $\mu_n(\alpha, \theta, q)$ in dependence on n , α and θ . The first result shows that the means are increasing in n .

Lemma 3.6. *For all $\alpha \geq 0$ $\mu_n(\alpha, \theta, q)$ is increasing in n .*

Proof. For $0 \leq x < y \leq n$ we have

$$q^{-x} < q^{-y}.$$

By elementary calculations, this can be written as

$$\frac{1}{1 - q^{n+1-x}} < \frac{1}{1 - q^{n+1-y}}.$$

This is equivalent to

$$\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q (y-x) < \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q (y-x)$$

and

$$\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q y + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q x < \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q x + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q y.$$

Multiplication with $\theta^{x+y} q^{\alpha(x^2+y^2)}$ yields

$$\begin{aligned} & \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q y \theta^{x+y} q^{\alpha(x^2+y^2)} + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q x \theta^{x+y} q^{\alpha(x^2+y^2)} \\ & < \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q x \theta^{x+y} q^{\alpha(x^2+y^2)} + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q y \theta^{x+y} q^{\alpha(x^2+y^2)}. \end{aligned}$$

Now we sum over all pairs (x, y) with $x < y$:

$$\sum_{\substack{x, y=0 \\ x \neq y}}^n \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \begin{bmatrix} n \\ y \end{bmatrix}_q y \theta^y q^{\alpha y^2} < \sum_{\substack{x, y=0 \\ x \neq y}}^n \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} x \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}.$$

By adding the terms for $x = y$ and an extra-sum we get

$$\begin{aligned} & \theta^{n+1} q^{\alpha(n+1)^2} \sum_{y=0}^n y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} + \sum_{x=0}^n \sum_{y=0}^n \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \begin{bmatrix} n \\ y \end{bmatrix}_q y \theta^y q^{\alpha y^2} \\ & < (n+1) \theta^{n+1} q^{\alpha(n+1)^2} \sum_{y=0}^n \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} + \sum_{x=0}^n \sum_{y=0}^n \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} x \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}. \end{aligned}$$

This can be written as

$$\sum_{x=0}^{n+1} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} < \sum_{x=0}^{n+1} x \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2},$$

and so we have

$$\frac{\sum_{y=0}^n y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}}{\sum_{y=0}^n \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}} < \frac{\sum_{x=0}^{n+1} x \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}{\sum_{x=0}^{n+1} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}. \quad \square$$

The means are increasing in the parameter θ too:

Lemma 3.7. *The means $\mu_n(\alpha, \theta, q)$ are increasing in θ for all $\alpha \geq 0$.*

Proof. We show that $\frac{\partial}{\partial \theta} \mu_n(\alpha, \theta, q) > 0$. Differentiating gives

$$\begin{aligned} & \frac{\partial}{\partial \theta} \left(\frac{\sum_{x=0}^n x \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}{\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}} \right) = \\ & = \frac{\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y^2 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^{y-1} q^{\alpha y^2} - \sum_{x=0}^n x \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^{y-1} q^{\alpha y^2}}{\left(\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \right)^2}. \end{aligned}$$

Thus it suffices to show that

$$\left(\sum_{x=1}^n x \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^{x-1} \right)^2 < \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^{x-1} q^{\alpha x^2} \sum_{y=0}^n y^2 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^{y-1} q^{\alpha y^2}.$$

The left-hand side can be written as

$$\sum_{x=1}^n x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q^2 q^{2\alpha x^2} \theta^{2(x-1)} + \sum_{\substack{x,y=0 \\ x \neq y}}^n xy \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} =: A_1 + B_1$$

and the right-hand side as

$$\sum_{x=1}^n x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q^2 q^{2\alpha x^2} \theta^{2(x-1)} + \sum_{\substack{x,y=0 \\ x \neq y}}^n x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} =: A_2 + B_2.$$

Since $A_1 = A_2$, it suffices to show that $B_1 < B_2$. For this purpose we consider the pairs (x, y) and (y, x) with $x < y$: In B_1 we have the term

$$(2) \quad 2xy \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2}$$

and in B_2

$$(3) \quad \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} (x^2 + y^2).$$

Since $2xy < x^2 + y^2$ for $x \neq y$, we have (2) < (3) and so $B_1 < B_2$. \square

For α the situation is a little bit different:

Lemma 3.8. $\mu_n(\alpha, \theta, q)$ is decreasing in α if $\alpha \in (0, 1]$ and increasing in α if $\alpha \geq 1$.

Proof. Assume $\alpha > 1$ (in the same way we can treat the case $0 < \alpha < 1$). We show that $\frac{\partial}{\partial \alpha} \mu_n(\alpha, \theta, q) > 0$. This is equivalent to

$$\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y^3 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} \log \alpha > \sum_{x=0}^n x \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y^2 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} \log \alpha.$$

So it is sufficient to show that

$$\sum_{\substack{x,y=0 \\ x \neq y}}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y} y^3 \log \alpha > \sum_{\substack{x,y=0 \\ x \neq y}}^n \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y} xy^2 \log \alpha.$$

Considering the pairs (x, y) and (y, x) , it is sufficient that $x^3 + y^3 > xy^2 + yx^2$. This is true because this can be written as $(y^2 - x^2)(y - x) = (y + x)(y - x)^2 > 0$. \square

Finally, a straightforward calculation shows that our family \mathcal{B} is closed under reversing, i.e., passing from X to $n - X$.

Theorem 3.9. If $X \sim \mathcal{B}(\alpha, \theta, n, q)$ then $n - X \sim \mathcal{B}(\alpha, \theta^{-1}q^{-2\alpha n}, n, q)$.

4. CONVERGENT PARAMETER

In this section we consider sequences $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ where the parameter sequence θ_n tends to a finite limit as $n \rightarrow \infty$. This will lead to the family \mathcal{P} as limit law. In particular we prove that the convergence of the classical binomial distribution with constant mean has a q -analogue. But in the case $\alpha = 0$ and $\theta_n \rightarrow 1$ these results fail. In this case we obtain — depending on the limit of θ_n — a uniform distribution or exponential-like distributions. In the following we need the two auxiliary results below.

Lemma 4.1. For $\alpha > 0$ we have for all $z \in \mathbb{C}$

$$\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} z^x \rightarrow E_q^{2\alpha}(z), \quad \text{as } n \rightarrow \infty.$$

For $\alpha = 0$ this holds for $|z| < 1$.

Proof. We estimate the difference

$$\begin{aligned} & \left| \sum_{x=0}^{\infty} \frac{q^{\alpha x^2}}{(q, q)_x} z^x - \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} z^x \right| \\ & \leq \sum_{x=n+1}^{\infty} \frac{q^{\alpha x^2}}{(q, q)_x} |z|^x + \sum_{x=1}^n q^{\alpha x^2} |z|^x \left| \begin{bmatrix} n \\ x \end{bmatrix}_q - \frac{1}{(q, q)_x} \right|. \end{aligned}$$

Estimating in the first sum the q -shifted factorial by the q -exponential function yields

$$\leq e_q(q) \sum_{x=n+1}^{\infty} (q^{\alpha n} |z|)^x + \sum_{x=1}^n \frac{q^{\alpha x^2}}{(q, q)_x} |z|^x \left(1 - \prod_{i=1}^x (1 - q^{n-i+1}) \right);$$

we use the same estimate for the second sum, split it and compute the first sum to obtain

$$\begin{aligned} \leq e_q(q) & \left(\frac{(q^{\alpha n} |z|)^{n+1}}{1 - q^{\alpha n} |z|} + \sum_{x=1}^{\lfloor \frac{n}{2} \rfloor} q^{\alpha x^2} |z|^x \left(1 - \prod_{i=1}^x (1 - q^{n-i+1}) \right) \right. \\ & \left. + \sum_{x=\lfloor \frac{n}{2} \rfloor}^n q^{\alpha x^2} |z|^x \left(1 - \prod_{i=1}^x (1 - q^{n-i+1}) \right) \right). \end{aligned}$$

The first term is obviously $o(1)$. Estimating the products gives

$$\leq e_q(q) \left(o(1) + \left(1 - \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (1 - q^{n-i}) \right) \sum_{x=1}^{\infty} q^{\alpha x^2} |z|^x + \sum_{x=\lfloor \frac{n}{2} \rfloor}^{\infty} q^{\alpha x^2} |z|^x \right)$$

and further

$$\leq e_q(q) \left(o(1) + \left(1 - \left(1 - q^{\lfloor \frac{n}{2} \rfloor} \right)^{\lfloor \frac{n}{2} \rfloor} \right) \sum_{x=1}^{\infty} q^{\alpha x^2} |z|^x + \sum_{x=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(q^{\alpha \lfloor \frac{n}{2} \rfloor} |z| \right)^x \right);$$

the latter sum is $o(1)$ as before, thus

$$= o(1) + O(n^2 q^n) \sum_{x=1}^{\infty} q^{\alpha x^2} |z|^x = o(1). \quad \square$$

Lemma 4.2. *Assume $\alpha > 0$ and let (θ_n) be a sequence of real numbers with limit $\theta \geq 0$. Then*

$$\lim_{n \rightarrow \infty} \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta_n^x = E_q^{2\alpha}(\theta).$$

If $\theta < 1$, this holds for $\alpha = 0$ as well.

Proof. For small $\epsilon > 0$ and n large enough we have

$$\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} (\theta - \epsilon)^x \leq \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta_n^x \leq \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} (\theta + \epsilon)^x,$$

hence, with use of Lemma 4.1,

$$\begin{aligned} E_q^{2\alpha}(\theta - \epsilon) &= \lim_{n \rightarrow \infty} \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} (\theta - \epsilon)^x \leq \liminf_{n \rightarrow \infty} \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta_n^x \\ &\leq \limsup_{n \rightarrow \infty} \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta_n^x \leq \lim_{n \rightarrow \infty} \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} (\theta + \epsilon)^x \\ &= E_q^{2\alpha}(\theta + \epsilon). \end{aligned}$$

By continuity of $E_q^{2\alpha}$, the lemma follows. \square

The first result is a generalisation of the fact that Kemp's distribution converges to the Heine distribution (see [12]).

Proposition 4.3. *If $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$, $\alpha > 0$, then for $n \rightarrow \infty$*

$$X_n \rightarrow \mathcal{P}(\alpha, \theta, q),$$

if $\theta_n \rightarrow \theta$. This still remains true in the case $\alpha = 0$ and $\theta < 1$.

Proof. This follows immediately from the fact that

$$\begin{bmatrix} n \\ x \end{bmatrix}_q \rightarrow \frac{1}{(q, q)_x}$$

for $n \rightarrow \infty$ and from Lemma 4.2. \square

In the case $\alpha = 0$ and $\theta > 1$ the situation is slightly different:

Proposition 4.4. *If $X_n \sim \mathcal{B}(0, \theta_n, n, q)$, then for $n \rightarrow \infty$, if $\theta_n \rightarrow \theta > 1$,*

$$n - X_n \rightarrow \mathcal{P}\left(0, \frac{1}{\theta}, q\right),$$

which is an Euler distribution.

Proof. Define $Y_n = n - X_n$. Then

$$\mathbb{P}(Y_n = x) = \frac{\binom{n}{x}_q \theta_n^{n-x}}{\sum_{y=0}^n \binom{n}{y}_q \theta_n^{n-y}} = \frac{\binom{n}{x}_q \theta_n^{-x}}{\sum_{y=0}^n \binom{n}{y}_q \theta_n^{-y}} \rightarrow \frac{\theta^{-x}}{(q, q)_x} \frac{1}{\sum_{y=0}^{\infty} \frac{1}{(q, q)_y} \theta^{-y}}$$

by Lemma 4.2. □

In particular we are interested in sequences X_n such that the limits $q \rightarrow 1$ and $n \rightarrow \infty$ can be exchanged. The propositions above immediately yield

Corollary 4.5. *For each $\alpha > 0$ let $X_n \sim \mathcal{B}(\alpha, \theta_n(q), n, q)$ with $\theta_n(q) \rightarrow \theta(q)$. Additionally assume that $\theta_n(q) \rightarrow \lambda/n$ and $\theta(q)/(1-q) \rightarrow \lambda$ as $q \rightarrow 1$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{B}(\alpha, \theta_n(q), n, q) & \xrightarrow{n \rightarrow \infty} & \mathcal{P}(\alpha, (1-q)\theta(q), q) \\ q \rightarrow 1 \downarrow & & \downarrow q \rightarrow 1 \\ B\left(n, \frac{\lambda}{n}\right) & \xrightarrow{n \rightarrow \infty} & P(\lambda) \end{array}$$

One very natural way to choose the parameter sequence is to set $\theta_n(q) = \frac{\lambda}{[n-\lambda]_q}$, $\lambda > 0$.

The convergence $\mathcal{B}(\alpha, \theta_n(q), n, q) \rightarrow \mathcal{P}(\alpha, (1-q)\theta(q), q)$ still remains true for $\alpha = 0$ if we require $(1-q)\theta(q) < 1$. Moreover, the commutative diagram remains correct for given $\lambda > 0$, if we restrict q to values greater than or equal to $\max(0, 1 - \frac{1}{\lambda})$.

The next result is a q -analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution.

Theorem 4.6. *Fix $\mu > 0$ and $\alpha > 0$. Consider a sequence of random variables $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ with parameter sequence $\theta_n = \theta_n(q, \mu)$ chosen such that the means μ_n of X_n are equal to μ . Then we have*

- (i) *The sequence X_n converges to the limit law $\mathcal{P}(\alpha, \theta, q)$, where θ is the limit of the sequence θ_n .*
- (ii) *As $q \rightarrow 1$, X_n tends to a binomial distribution with parameters n and μ/n .*
- (iii) *In the limit $q \rightarrow 1$, $\mathcal{P}(\alpha, \theta(q, \mu), q)$ converges to a Poisson distribution with parameter μ .*

Thus the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B}(\alpha, \theta_n(q, \mu), n, q) & \xrightarrow{n \rightarrow \infty} & \mathcal{P}(\alpha, \theta(q, \mu), q) \\ q \rightarrow 1 \downarrow & & \downarrow q \rightarrow 1 \\ B\left(n, \frac{\mu}{n}\right) & \xrightarrow{n \rightarrow \infty} & P(\mu) \end{array}$$

Proof. First we check, that for given μ, q and large n there is a unique $\theta_n(q)$, such that $\mu_n(\theta_n(q), q) = \mu$. The function $\mu_n(\theta, q)$ is continuous and increasing in θ (see Lemma 3.7). Moreover $\lim_{\theta \rightarrow 0} \mu_n(\theta, q) = 0$. From Corollary 5.1 (ii) we see that for sufficiently large n and suitable θ_n , $\mu_n(\theta_n, q) \geq \frac{n}{2}$: Using the notation of Corollary 5.1, we deduce from the fact that $n - X_n$ converges to the point measure in 0, that the means of $n - X_n$ converge to 0; thus the means of X_n become arbitrarily large. Consequently there exists a unique solution $\theta_n(q)$ of $\mu_n(\theta, q) = \mu$. By [15, Lemma 3.3], $\theta_n(q)$ converges to a limit $\theta(q)$, where $\theta(q)$ is the unique solution of $\mu_\infty(\theta, q) = \mu$. Hence $\mathcal{B}(\alpha, \theta_n(q), n, q) \rightarrow \mathcal{P}(\alpha, \theta(q), q)$ by Lemma 4.2.

Again by [15, Lemma 3.3] we get that $\theta_n(q) \rightarrow \frac{\mu}{n-\mu}$ for $q \rightarrow 1$ and so $\frac{\theta_n(q)}{1+\theta_n(q)} \rightarrow \frac{\mu}{n}$. Consequently $\mathcal{B}(\alpha, \theta_n(q), n, q) \rightarrow B\left(n, \frac{\mu}{n}\right)$.

It remains to check that $\theta(q)/(1-q)$ converges to μ for $q \rightarrow 1$ (then $\mathcal{P}(\alpha, \theta(q), q) \rightarrow P(\mu)$). The value $\theta(q)/(1-q)$ is the unique solution of $\mu_\infty((1-q)\theta, q) = \mu$. Moreover, $\mu_\infty((1-q)\theta, q)$ converges pointwise to θ for $q \rightarrow 1$, so we can apply [15, Lemma 3.3]. □

In the case $\alpha = 0$ an analogous result holds for X_n or $n - X_n$ depending on the values of the parameters, i.e., if $\theta(q, \mu) < 1$ then the theorem holds for the sequence X_n , and if $\theta(q, \mu) > 1$ then this is true for $n - X_n$.

Now we turn our attention to the case $\alpha = 0$. To finish the analysis of the RS-distribution we consider $\theta_n \rightarrow 1$. It is worthwhile to point out that the limit distributions only depend on the growth rate of the parameter sequences and are independent of q . This is why we will distinguish three cases in dependence on the speed of the convergence of the parameters θ_n to the limit 1. First we will provide a result of fast growing θ_n . In order to do so we start with an auxiliary result.

Lemma 4.7. *If $f(n) \leq n$, $\theta_n \leq 1$ and $f(n) \rightarrow \infty$ and $\theta_n^{f(n)} \rightarrow 1$ for $n \rightarrow \infty$, then for $k \in \mathbb{N}$*

$$\sum_{0 \leq i \leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i \sim e_q(q) \frac{f(n)^{k+1}}{k+1}, \quad n \rightarrow \infty.$$

Proof. Write

$$\begin{aligned} \sum_{0 \leq i \leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i &= \sum_{i=0}^{\lfloor \sqrt{f(n)} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i + \sum_{i=\lfloor \sqrt{f(n)} \rfloor+1}^{f(n)-\lfloor \sqrt{f(n)} \rfloor-1} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i \\ &\quad + \sum_{f(n)-\lfloor \sqrt{f(n)} \rfloor \leq i \leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i. \end{aligned}$$

The first and the third term on the right-hand side can be estimated by

$$(\sqrt{f(n)} + 1) f(n)^k \frac{(q, q)_n}{(q, q)_{\lfloor n/2 \rfloor} (q, q)_{n-\lfloor n/2 \rfloor}}$$

and are therefore negligible. The middle term can be bounded by

$$\begin{aligned} \frac{(q, q)_n}{(q, q)_{\lfloor \sqrt{f(n)} \rfloor+1} (q, q)_{n-\lfloor \sqrt{f(n)} \rfloor-1}} \theta_n^{f(n)} \sum_{i=\lfloor \sqrt{f(n)} \rfloor+1}^{f(n)-\lfloor \sqrt{f(n)} \rfloor-1} i^k &\leq \sum_{i=\lfloor \sqrt{f(n)} \rfloor+1}^{f(n)-\lfloor \sqrt{f(n)} \rfloor-1} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i \\ &\leq \frac{(q, q)_n}{(q, q)_{\lfloor n/2 \rfloor} (q, q)_{n-\lfloor n/2 \rfloor}} \sum_{i=\lfloor \sqrt{f(n)} \rfloor+1}^{f(n)-\lfloor \sqrt{f(n)} \rfloor-1} i^k \end{aligned}$$

and has the asserted asymptotic. \square

This lemma implies that under the assumption $\theta_n^n \rightarrow 1$ the limit law is uniform on the interval $[-\sqrt{3}, \sqrt{3}]$.

Theorem 4.8. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \leq 1$ and $\theta_n^n \rightarrow 1$, then $(X_n - \mu_n)/\sigma_n$ converges for $n \rightarrow \infty$ to the uniform distribution on the interval $[-\sqrt{3}, \sqrt{3}]$.*

Proof. We start with an asymptotic of the means and the variances. By Lemma 4.7 we have

$$\mu_n = \frac{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i \theta_n^i}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} \sim \frac{e_q(q) \frac{n^2}{2}}{e_q(q) n} = \frac{n}{2}$$

and

$$\sigma_n^2 = \frac{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i^2 \theta_n^i}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} - \mu_n^2 \sim \frac{n^2}{3} - \frac{n^2}{4} = \frac{n^2}{12}.$$

From these two facts one can easily see that the support of the limiting distribution is

$$\lim_{n \rightarrow \infty} [-\mu_n/\sigma_n, (n - \mu_n)/\sigma_n] = [-\sqrt{3}, \sqrt{3}].$$

Now we compute

$$\begin{aligned} \mathbb{P}(X \leq x) &= \lim_{n \rightarrow \infty} \sum_{-\frac{\mu_n}{\sigma_n} \leq \frac{k - \mu_n}{\sigma_n} \leq x} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} \sum_{0 \leq k \leq \sigma_n x + \mu_n} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k \\ &= \lim_{n \rightarrow \infty} \frac{1}{e_q(q) n} e_q(q) (\sigma_n x + \mu_n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{n}{2\sqrt{3}} x + \frac{n}{2} \right) = \frac{1}{2\sqrt{3}} x + \frac{1}{2}, \end{aligned}$$

which is the distribution function of the uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. \square

Using the fact that a $RS(n, \theta, q)$ -distribution corresponds to an $(n - RS(n, 1/\theta, q))$ -distribution or following the above proofs we immediately get the following corollary:

Corollary 4.9. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \geq 1$ and $\theta_n^n \rightarrow 1$, then $(\mu_n - X_n)/\sigma_n$ and $(X_n - \mu_n)/\sigma_n$ converge for $n \rightarrow \infty$ to the uniform distribution on the interval $[-\sqrt{3}, \sqrt{3}]$.*

Now we turn to the case that $\theta_n^n \rightarrow c$ with $0 < c < 1$. For this purpose we start with the following lemma, which supplements [15, Lemmas 4.4 and 4.5] and is crucial for the analysis of the variances.

Lemma 4.10. *For $\theta_n \leq 1$ and $\theta_n \rightarrow 1$, $\theta_n^n \rightarrow c$ with $0 < c < 1$ and $f(n)/n \sim \beta > 0$ we have*

$$\sum_{i=0}^n i^2 \theta_n^i \sim -2 \frac{1-c + c \log c - \frac{1}{2} c \log^2 c}{\log^3 c} n^3$$

as $n \rightarrow \infty$.

$$\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i^2 \theta_n^i \sim -2e_q(q) \frac{1-c + c \log c - \frac{1}{2} c \log^2 c}{\log^3 c} n^3$$

as $n \rightarrow \infty$.

Proof. Using

$$\sum_{i=0}^n i^2 t^i = \frac{t(-1-t+t^n + 2nt^n(1-t) + n^2t^n(1-t)^2 + t^{n+1})}{(t-1)^3}$$

we obtain for the first sum

$$\sum_{i=0}^n i^2 \theta_n^i \sim (-2 + 2c - 2c \log c + c \log^2 c) \frac{n^3}{\log^3 c}.$$

The second sum follows immediately as in [15, Lemma 4.4]. \square

Now we are able to establish the convergence result in this case.

Theorem 4.11. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \leq 1$, $\theta_n \rightarrow 1$ and $\theta_n^n \rightarrow c$ with $0 < c < 1$, then $(X_n - \mu_n)/\sigma_n$ converges to a limit distribution X with*

$$\mathbb{P}(X \leq x) = \frac{c^{\alpha(c,x)} - 1}{c - 1},$$

where

$$\alpha(c, x) = \frac{\sqrt{(c-1)^2 - c \log^2 c}}{(c-1) \log c} x + \frac{1-c + c \log c}{(c-1) \log c}$$

and $x \in [-\gamma_1, \gamma_2]$ with

$$\gamma_1 = \frac{1-c + c \log c}{\sqrt{(c-1)^2 - c \log^2 c}} \quad \text{and} \quad \gamma_2 = \frac{c-1 - \log c}{\sqrt{(c-1)^2 - c \log^2 c}}.$$

Proof. Using [15, Lemmas 4.4 and 4.5] and Lemma 4.10 we get for the means μ_n

$$\mu_n = \frac{\sum_{i=0}^n i \theta_n^i \begin{bmatrix} n \\ i \end{bmatrix}_q}{\sum_{i=0}^n \theta_n^i \begin{bmatrix} n \\ i \end{bmatrix}_q} \sim \frac{(1-c + c \log c) n^2}{\log^2 c} \frac{\log c}{(c-1)n} = \frac{1-c + c \log c}{(c-1) \log c} n$$

and for the variances σ_n^2

$$\begin{aligned} \sigma_n^2 &= \frac{\sum_{i=0}^n i^2 \theta_n^i \begin{bmatrix} n \\ i \end{bmatrix}_q}{\sum_{i=0}^n \theta_n^i \begin{bmatrix} n \\ i \end{bmatrix}_q} - \mu_n^2 \\ &\sim \frac{-2(1-c + c \log c - \frac{1}{2} c \log^2 c) n^3}{\log^3 c} \frac{\log c}{(c-1)n} - \frac{(1-c + c \log c)^2}{(c-1)^2 \log^2 c} n^2 \\ &= \frac{c^2 + 1 - 2c - c \log^2 c}{(c-1)^2 \log^2 c} n^2. \end{aligned}$$

As an immediate consequence we get that the support of the limit distribution

$$[-\gamma_1, \gamma_2] = \lim_{n \rightarrow \infty} [-\mu_n/\sigma_n, (n - \mu_n)/\sigma_n]$$

is as stated in the theorem. Now we compute the distribution function of X :

$$\mathbb{P}(X \leq x) = \lim_{n \rightarrow \infty} \sum_{\frac{k - \mu_n}{\sigma_n} \leq x} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} = \lim_{n \rightarrow \infty} \frac{1}{e_q(q) \frac{c-1}{\log c} n} \sum_{k \leq \sigma_n x + \mu_n} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k.$$

Since $\sigma_n x + \mu_n \sim n\alpha(c, x)$ we have further

$$\begin{aligned} \mathbb{P}(X \leq x) &= \lim_{n \rightarrow \infty} \frac{1}{e_q(q)^{\frac{c-1}{\log c} n}} \sum_{k \leq n\alpha(c, x)} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^q \\ &= \lim_{n \rightarrow \infty} \frac{c^{\alpha(c, x)} - 1}{e_q(q)^{\frac{c-1}{\log c} n}} = \frac{c^{\alpha(c, x)} - 1}{c - 1}, \end{aligned}$$

what completes the proof of this theorem. \square

Again we get the following immediate consequence:

Corollary 4.12. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \geq 1$, $\theta_n \rightarrow 1$ and $\theta_n \rightarrow \tilde{c}$ with $1 < \tilde{c} < \infty$, then $(\mu_n - X_n)/\sigma_n$ and $(X_n - \mu_n)/\sigma_n$ converge to a limit X , whose distribution is given in Theorem 4.11 with $c = 1/\tilde{c}$ resp. \tilde{c} .*

Finally we study the case that $\theta_n^{f(n)} \rightarrow c$ with $0 < c < 1$ and $f(n) = o(n)$. The analysis of this case is very similar to that of the previous case. So we start again with a lemma which is useful to find the asymptotic behaviour of the means and variances.

Lemma 4.13. *Let $f(n) \rightarrow \infty$ for $n \rightarrow \infty$, $\frac{f(n)}{n} \rightarrow 0$, $\theta_n^{f(n)} \rightarrow c$ with $0 < c < 1$ and. Then*

$$\sum_{i=0}^n i^2 \theta_n^i \sim \frac{f(n)^3}{\log^3 c}$$

and

$$\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i^2 \theta_n^i \sim e_q(q) \frac{f(n)^3}{\log^3 c}.$$

Proof. Similar to [15, Lemma 4.8]. \square

The following theorem shows that in this case the limiting distribution is an exponential distribution.

Theorem 4.14. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \leq 1$, $\theta_n \rightarrow 1$, $\theta_n^{f(n)} \rightarrow c$ with $f(n) = o(n)$ and $0 < c < 1$, then $(X_n - \mu_n)/\sigma_n$ converges to a normalised exponential distribution with parameter $\lambda = 1$, i.e.,*

$$\mathbb{P}(X \leq x) = 1 - e^{-x-1}, \quad x \geq -1.$$

Proof. From [15, Lemmas 4.4 and 4.8] and Lemma 4.13 we get

$$\mu_n \sim \frac{-f(n)}{\log c} \quad \text{and} \quad \sigma_n^2 \sim \frac{2f(n)^2}{\log^2 c} - \frac{f(n)^2}{\log^2 c} = \frac{f(n)^2}{\log^2 c}.$$

Computing the distribution function of the limit distribution yields

$$\begin{aligned} \mathbb{P}(X \leq x) &= \lim_{n \rightarrow \infty} \sum_{k \leq \sigma_n x + \mu_n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\frac{-e_q(q)}{\log c} n} \sum_{k \leq \frac{-f(n)}{\log c} x + \frac{-f(n)}{\log c}} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k \\ &= \lim_{n \rightarrow \infty} \frac{1 - c^{\frac{1+x}{-\log c}}}{\log c} f(n) e_q(q) \frac{\log c}{e_q(q) f(n)} \\ &= 1 - c^{\frac{1+x}{-\log c}} = 1 - e^{-x-1}. \end{aligned} \quad \square$$

Corollary 4.15. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \geq 1$, $\theta_n \rightarrow 1$, $\theta_n^{f(n)} \rightarrow c$ with $f(n) = o(n)$ and $1 < c < \infty$, then $(\mu_n - X_n)/\sigma_n$ converges to a normalised exponential distribution with parameter $\lambda = 1$.*

5. UNBOUNDED PARAMETER

Now we turn our attention to sequences of random variables X_n with $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$, where the parameter sequence $\theta_n = \theta_n(q)$ tends to infinity. We start with fast growing parameters θ_n , i.e., $\theta_n = q^{-2\alpha n - g(n)}$ with $g(n)$ convergent or $g(n) \rightarrow \infty$. Due to the reversing property 3.9 we conclude immediately from Lemma 4.2:

Corollary 5.1. *Let $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ with $\theta_n = q^{-2\alpha n - g(n)}$.*

- (i) *If $g(n) \rightarrow \gamma$ then for $\alpha > 0$ we have $n - X_n \rightarrow \mathcal{P}(\alpha, q^{-\gamma}, q)$.*
- (ii) *If $g(n) \rightarrow \infty$ then for all $\alpha \geq 0$ we have $n - X_n \rightarrow \delta_0$.*

Now we consider parameter sequences $\theta_n(q) = q^{-f(n)}$ with $f(n) \rightarrow \infty$ and $2\alpha n - f(n) \rightarrow \infty$ for $n \rightarrow \infty$ and $\alpha > 0$. These assumptions will be in force throughout the section. We will prove in Theorem 5.7 that a suitably chosen subsequence of the normalised sequence of random variables X_n converges to a discrete normal distribution. Lemmas 5.2, 5.3, and 5.4 are devoted to the asymptotic behaviour of the sequence (μ_n) of means. Afterwards we study the sequence (σ_n) of variances in Lemmas 5.5 and 5.6 and then we establish the convergence result.

To simplify notation, we define

$$\begin{aligned}\Sigma_1(z) &:= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} z \left[\begin{matrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{matrix} \right]_q q^{\alpha(a+x)^2}, & \Sigma_1 &:= \Sigma_1(1), \\ \Sigma_2(z) &:= \sum_{x=0}^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor - 1} z \left[\begin{matrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{matrix} \right]_q q^{\alpha(a-(x+1))^2}, & \Sigma_2 &:= \Sigma_2(1), \\ \Sigma_1^\infty(z) &:= \sum_{x=0}^{\infty} z q^{\alpha(a+x)^2}, & \Sigma_1^\infty &:= \Sigma_1^\infty(1), \\ \Sigma_2^\infty(z) &:= \sum_{x=0}^{\infty} z q^{\alpha(a-(x+1))^2}, & \Sigma_2^\infty &:= \Sigma_2^\infty(1),\end{aligned}$$

where $a = \left\{ \frac{f(n)}{2\alpha} \right\}$.

Now we turn to the study of the sequence of means.

Lemma 5.2. *For $n \rightarrow \infty$ we have*

$$\mu_n = \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + c(a, \alpha, q) + o(1),$$

where

$$c(a, \alpha, q) = \frac{\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^{\infty} \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2} \right)}.$$

Proof. We have to study the behaviour of

$$\frac{\sum_{x=0}^n x \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}{\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}.$$

For this purpose we expand the fraction by $q^{\frac{f(n)^2}{4\alpha}}$ and analyse the denominator D and the numerator N separately.

$$D = \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}} = \sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\frac{(-2\alpha x + f(n))^2}{4\alpha}};$$

splitting the sum into two parts gives

$$= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\frac{(-2\alpha x + f(n))^2}{4\alpha}} + \sum_{x=\lfloor \frac{f(n)}{2\alpha} \rfloor + 1}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\frac{(-2\alpha x + f(n))^2}{4\alpha}}.$$

By reversing the order of summation in the first sum and shifting the summation index in the second sum we obtain

$$\begin{aligned}&= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} \left[\begin{matrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{matrix} \right]_q q^{\frac{(-2\alpha \lfloor \frac{f(n)}{2\alpha} \rfloor + f(n) + 2\alpha x)^2}{4\alpha}} \\ &\quad + \sum_{x=0}^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor - 1} \left[\begin{matrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{matrix} \right]_q q^{\frac{(-2\alpha \lfloor \frac{f(n)}{2\alpha} \rfloor - 2\alpha x + f(n))^2}{4\alpha}};\end{aligned}$$

simplifying the exponents of q leads to

$$\begin{aligned}
 &= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} \left[\begin{matrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{matrix} \right]_q q^{\alpha(a+x)^2} \\
 &\quad + \sum_{x=0}^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor - 1} \left[\begin{matrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{matrix} \right]_q q^{\alpha(a-(x+1))^2}.
 \end{aligned}$$

This tends to

$$(4) \quad e_q(q) \left(\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=0}^{\infty} q^{\alpha(a-(x+1))^2} \right) =: \gamma$$

since we can bound the first sum as follows:

$$\begin{aligned}
 e_q(q) \sum_{x=0}^{\infty} q^{\alpha(a+x)^2} &\geq \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} \left[\begin{matrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{matrix} \right]_q q^{\alpha(a+x)^2} \\
 &\geq \sum_{x=0}^{\frac{1}{2} \lfloor \frac{f(n)}{2\alpha} \rfloor} \left[\begin{matrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{matrix} \right]_q q^{\alpha(a+x)^2},
 \end{aligned}$$

estimating the q -binomial coefficient yields

$$\begin{aligned}
 &\geq \frac{\left(1 - q^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor + 1}\right)^{\lfloor \frac{f(n)}{2\alpha} \rfloor + 1}}{(q, q)_{\frac{1}{2} \lfloor \frac{f(n)}{2\alpha} \rfloor}} \sum_{x=0}^{\frac{1}{2} \lfloor \frac{f(n)}{2\alpha} \rfloor} q^{\alpha(a+x)^2} \\
 &\rightarrow e_q(q) \sum_{x=0}^{\infty} q^{\alpha(a+x)^2}.
 \end{aligned}$$

Here we used that

$$\left(1 - q^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor + 1}\right)^{\lfloor \frac{f(n)}{2\alpha} \rfloor + 1} = 1 + \mathcal{O}\left(\left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + 1\right) n q^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor + 1}\right).$$

Similar arguments hold for the second sum. Now we turn to the numerator N .

$$N = \sum_{x=0}^n x \left[\begin{matrix} n \\ x \end{matrix} \right]_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}},$$

we split the sum again, reverse the order of summation resp. shift the summation index and get

$$\begin{aligned}
 &= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} \left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - x \right) \left[\begin{matrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{matrix} \right]_q q^{\alpha(a+x)^2} + \\
 &\quad + \sum_{x=0}^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor - 1} \left(x + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + 1 \right) \left[\begin{matrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{matrix} \right]_q q^{\alpha(a-(x+1))^2}.
 \end{aligned}$$

Using the same arguments as above yields

$$(5) \quad = \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \gamma - e_q(q) (\Sigma_1^\infty(x) - \Sigma_2^\infty(x)) + e_q(q) \Sigma_2^\infty + o(1).$$

Combining (4) and (5) we obtain

$$\mu_n = \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + \frac{\sum_{x=0}^{\infty} q^{\alpha(a-(x+1))^2} - \sum_{x=0}^{\infty} x \left(q^{\alpha(a+x)^2} - q^{\alpha(a-(x+1))^2} \right)}{\sum_{x=0}^{\infty} \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2} \right)} + o(1).$$

Simplifying the fraction yields the theorem. \square

Now we provide an estimate for the $O(1)$ -term in the preceding lemma.

Lemma 5.3. *Let $c(a, \alpha, q)$ be defined as in Lemma 5.2. Then*

- (i) $0 \leq c(a, \alpha, q) < 1$,
- (ii) $c(a, \alpha, q) = 0 \Leftrightarrow a = 0$,
- (iii) $c(a, \alpha, q) + c(1 - a, \alpha, q) = 1$.

Proof. Since for all $x \geq 0$

$$(6) \quad q^{\alpha(-a+x)^2} \geq q^{\alpha(a+x)^2},$$

$0 \leq c(a, \alpha, q)$. Moreover, $c(a, \alpha, q) = 0$ iff in (6) equality holds for all $x \geq 1$. But this is the case iff $(x-a)^2 = (x+a)^2$ for all x . So $c(a, \alpha, q) = 0$ iff $a = 0$. For (i) it remains to show that

$$\sum_{x=1}^{\infty} x \left(q^{\alpha(-a+x)^2} - q^{\alpha(a+x)^2} \right) < \sum_{x=1}^{\infty} \left(q^{\alpha(-a+x)^2} + q^{\alpha(a+x)^2} \right) + q^{\alpha a^2}.$$

We can rewrite this as

$$\sum_{x=1}^{\infty} (x-1)q^{\alpha(x-a)^2} - \sum_{x=0}^{\infty} (x+1)q^{\alpha(x+a)^2} < 0.$$

The left-hand side is increasing in a , and for $a = 1$ we have

$$\sum_{x=1}^{\infty} (x-1)q^{\alpha(x-1)^2} - \sum_{x=0}^{\infty} (x+1)q^{\alpha(x+1)^2} = 0.$$

Since $0 \leq a < 1$, (i) follows.

To see (iii), note that the denominators of $c(a, \alpha, q)$ and $c(1-a, \alpha, q)$ are invariant under the substitution $a \mapsto 1-a$. Then add the numerators:

$$\sum_{x=1}^{\infty} x \left(q^{\alpha(1-a-x)^2} - q^{\alpha(1-a+x)^2} \right) + \sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2} \right),$$

by splitting the first sum and shifting the summation index we obtain

$$\begin{aligned} &= \sum_{x=0}^{\infty} (x+1)q^{\alpha(a+x)^2} - \sum_{x=2}^{\infty} (x-1)q^{\alpha(a-x)^2} + \sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2} \right) \\ &= q^{\alpha a^2} + \sum_{x=1}^{\infty} q^{\alpha(a+x)^2} + q^{\alpha(a-1)^2} + \sum_{x=2}^{\infty} q^{\alpha(a-x)^2} \\ &= \sum_{x=0}^{\infty} \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2} \right), \end{aligned}$$

which is exactly the denominator of $c(a, \alpha, q)$. □

Lemma 5.4. *Let $c(a, \alpha, q)$ be defined as in Lemma 5.2.*

- (i) *If $a > 0$, then $\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + c(a, \alpha, q) \notin \mathbb{Z}$.*
- (ii) *If $a = 0$, then*

$$\mu_n \begin{cases} \geq \frac{f(n)}{2\alpha} & \text{if } n \geq \frac{f(n)}{\alpha} \\ < \frac{f(n)}{2\alpha} & \text{if } n < \frac{f(n)}{\alpha} \end{cases}.$$

Proof. Lemma 5.3 implies (i). To see (ii) we use

$$\mu_n = \frac{f(n)}{2\alpha} + \frac{\sum_{x=1}^{n-\frac{f(n)}{2\alpha}} x \left[x + \frac{n}{2\alpha} \right]_q q^{\alpha x^2} - \sum_{x=1}^{\frac{f(n)}{2\alpha}} x \left[\frac{f(n)}{2\alpha} - x \right]_q q^{\alpha x^2}}{\sum_{x=0}^{\frac{f(n)}{2\alpha}} \left[\frac{f(n)}{2\alpha} - x \right]_q q^{\alpha x^2} + \sum_{x=1}^{n-\frac{f(n)}{2\alpha}} \left[x + \frac{n}{2\alpha} \right]_q q^{\alpha x^2}}.$$

Now consider the case $n \geq \frac{f(n)}{\alpha}$: We have to prove that

$$\sum_{x=1}^{n-\frac{f(n)}{2\alpha}} x \left[x + \frac{n}{2\alpha} \right]_q q^{\alpha x^2} \geq \sum_{x=1}^{\frac{f(n)}{2\alpha}} x \left[\frac{f(n)}{2\alpha} - x \right]_q q^{\alpha x^2}.$$

We will see that for all $1 \leq x \leq \frac{f(n)}{2\alpha}$, the term on the right-hand side is less than or equal to the corresponding term on the left-hand side (there are enough terms on the left-hand side by our assumption), i.e.,

$$\left[x + \frac{n}{2\alpha} \right]_q \geq \left[\frac{f(n)}{2\alpha} - x \right]_q :$$

Our assumption implies

$$n - x - \frac{f(n)}{2\alpha} + 1 + i \geq \frac{f(n)}{2\alpha} - x + 1 + i \quad \text{for } 0 \leq i \leq 2x - 1.$$

Taking the product over all i on both sides yields

$$\left(1 - q^{n-x-\frac{f(n)}{2\alpha}+1}\right) \cdots \left(1 - q^{n-\frac{f(n)}{2\alpha}+x}\right) \geq \left(1 - q^{\frac{f(n)}{2\alpha}-x+1}\right) \cdots \left(1 - q^{x+\frac{f(n)}{2\alpha}}\right)$$

and therefore

$$\frac{1}{(q, q)_{x+\frac{f(n)}{2\alpha}} (q, q)_{n-x-\frac{f(n)}{2\alpha}}} \geq \frac{1}{(q, q)_{\frac{f(n)}{2\alpha}-x} (q, q)_{n-\frac{f(n)}{2\alpha}+x}},$$

and this leads to the assertion in this case immediately.

The case $n < \frac{f(n)}{\alpha}$ can be treated similarly. \square

In [6] Gerhold and Zeiner studied the behaviour of the means of Kemp's q -binomial distribution in the limit $q \rightarrow 0$ and $c(a, \alpha, q)$ in the limit $q \rightarrow 1$. We will do the same analysis here. First we will show that for $q \rightarrow 0$

$$\mu_n \rightarrow \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + \begin{cases} 0 & \text{if } 0 \leq a < \frac{1}{2} \\ \frac{1}{2} & \text{if } a = \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < a < 1 \end{cases}.$$

For this purpose we estimate $c(a, \alpha, q)$:

$$\begin{aligned} c(a, \alpha, q) &= \frac{q^{\alpha(1-a)^2} + 2q^{\alpha(2-a)^2} + \sum_{x=3}^{\infty} xq^{\alpha(a-x)^2} - \sum_{x=1}^{\infty} xq^{\alpha(a+x)^2}}{q^{\alpha a^2} + q^{\alpha(a-1)^2} + \sum_{x=1}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=2}^{\infty} q^{\alpha(a-x)^2}} \\ &\leq \frac{q^{\alpha(1-a)^2} + 2q^{\alpha(2-a)^2} + \sum_{x=3}^{\infty} xq^{\alpha(1-x)^2} - \sum_{x=1}^{\infty} xq^{\alpha(1+x)^2}}{q^{\alpha a^2} + q^{\alpha(a-1)^2} + \sum_{x=1}^{\infty} q^{\alpha(1+x)^2} + \sum_{x=2}^{\infty} q^{\alpha x^2}} \\ &= \frac{q^{\alpha(1-a)^2} + 2q^{\alpha(2-a)^2} + 2\sum_{x=2}^{\infty} q^{\alpha x^2}}{q^{\alpha a^2} + q^{\alpha(a-1)^2} + 2\sum_{x=2}^{\infty} q^{\alpha x^2}} \\ &= \frac{1 + 2q^{\alpha(3-2a)} + 2q^{-\alpha(1-a)^2} \sum_{x=2}^{\infty} q^{\alpha x^2}}{q^{\alpha(-1+2a)} + 1 + 2q^{-\alpha(1-a)^2} \sum_{x=2}^{\infty} q^{\alpha x^2}}. \end{aligned}$$

For $a \in [0, \frac{1}{2})$ we have $2a - 1 < 0$ and therefore the denominator tends to infinity while the numerator goes to 1. Consequently $c(a, \alpha, q) \rightarrow 0$. Lemma 5.3 (iii) implies that $c(a, \alpha, q) \rightarrow 1$ if $a \in (\frac{1}{2}, 1)$ and $c(\frac{1}{2}, \alpha, q) = \frac{1}{2}$. Moreover, from the estimates in the proof of Lemma 5.2 we get easily that the $o(1)$ -term vanishes in the limit $q \rightarrow 0$.

In the limit $q \rightarrow 1$ we have $c(a, \alpha, q) \rightarrow a$. To see this, apply the Euler-Maclaurin formula (see [1]) to

$$f^+(x) = q^{Ax^2+Bx} \quad \text{and} \quad g^+(x) = xq^{Ax^2+Bx}$$

first, which yields

$$\sum_{\ell \geq 0} f^+(\ell) = I_f^+ + \frac{f^+(0)}{2} + R_f^+$$

with

$$I_f^+ = \int_0^{\infty} f^+(x) dx \quad \text{and} \quad R_f^+ = \int_0^{\infty} \left(x - [x] - \frac{1}{2}\right) f^{+'}(x) dx.$$

Computing I_f^+ gives

$$I_f^+ = \frac{\sqrt{\pi} q^{-\frac{B^2}{4A}} \left(1 + \operatorname{erf}\left(\frac{B \log q}{2\sqrt{-A \log q}}\right)\right)}{2\sqrt{-A \log q}},$$

where $\operatorname{erf}(z)$ denotes the error-function. Similarly, we get for $g^+(x)$

$$\sum_{\ell \geq 0} g^+(\ell) = I_g^+ + \frac{g^+(0)}{2} + R_g^+$$

with

$$I_g^+ = \int_0^{\infty} g^+(x) dx \quad \text{and} \quad R_g^+ = \int_0^{\infty} \left(x - [x] - \frac{1}{2}\right) g^{+'}(x) dx.$$

Computing I_g^+ gives

$$I_g^+ = \frac{-B\sqrt{\pi} q^{-\frac{B^2}{4A}} \left(1 + \operatorname{erf}\left(\frac{B \log q}{2\sqrt{-A \log q}}\right)\right)}{4A\sqrt{-A \log q}} - \frac{1}{2} \frac{1}{A \log q}.$$

In an analogous way we treat the functions

$$f^-(x) = q^{Ax^2-Bx} \quad \text{and} \quad g^-(x) = xq^{Ax^2-Bx}.$$

Note that $f^-(0) = f^+(0) = \frac{1}{2}$ and $g^-(0) = g^+(0) = 0$. Putting things together we obtain with $A = \alpha$ and $B = 2\alpha\alpha$

$$\begin{aligned} c(a, \alpha, q) &= \frac{I_g^- + R_g^- - I_g^+ - R_g^+}{I_f^+ + I_f^- + R_f^+ + R_f^- + q^{-a^2}} \\ &= \frac{\frac{B\sqrt{\pi}q^{-\frac{B^2}{4A}}}{2A\sqrt{-A\log q}} + R_g^- - R_g^+}{\frac{\sqrt{\pi}q^{-\frac{B^2}{4A}}}{\sqrt{-A\log q}} + R_f^- + R_f^+ + q^{-a^2}} \\ &= \frac{a + q^{\frac{B^2}{4A}} \frac{\sqrt{-A\log q}}{\sqrt{\pi}} (R_g^- - R_g^+)}{1 + q^{\frac{B^2}{4A}} \frac{\sqrt{-A\log q}}{\sqrt{\pi}} (R_f^- + R_f^+ + q^{-a^2})}. \end{aligned}$$

Thus it remains to show that $\sqrt{-\log q}(R_g^- - R_g^+)$ and $\sqrt{-\log q}(R_f^- + R_f^+)$ both tend to 0. We have

$$R_f^+ = \int_0^\infty \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} \log q (2Ax + B) dx.$$

The integral

$$J_1 := B \int_0^\infty \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} dx$$

is bounded uniformly for all $q \in [0, 1)$, since q^{Ax^2+Bx} is decreasing in x :

$$\begin{aligned} -\frac{1}{4} &= -\frac{1}{4} + \sum_{i=0}^\infty 0 \\ &\leq \int_0^{\frac{1}{2}} \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} dx + \sum_{i=0}^\infty \int_{\frac{1}{2}+i}^{\frac{1}{2}+i+1} \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} dx \\ &= J_1 \\ &= \int_0^1 \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} dx + \sum_{i=1}^\infty \int_i^{i+1} \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} dx \\ &\leq 0 + \sum_{i=1}^\infty 0 = 0. \end{aligned}$$

Thus $(-\log q)^{3/2} J_1 \rightarrow 0$. With the same idea we want to estimate

$$J_2 := 2A \int_0^\infty \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} x dx.$$

Unfortunately $h(x) := q^{Ax^2+Bx} x$ need not be decreasing in x for $x \geq 0$. Differentiating gives

$$h'(x) = q^{Ax^2+Bx} (1 + (2Ax^2 + Bx) \log q).$$

Obviously $h'(0) > 0$ and $\lim_{x \rightarrow -\infty} h'(x) = \lim_{x \rightarrow \infty} h'(x) = -\infty$ since $\log q < 0$. Consequently there exists a single positive root r of $h'(x)$. For q near at 1 we have $r \leq 1/\sqrt{-A\log q}$ since

$$\begin{aligned} h' \left(\frac{1}{\sqrt{-A\log q}} \right) &\approx 1 + \log q \left(-\frac{2A}{A\log q} + \frac{B}{\sqrt{-A\log q}} \right) \\ &= 1 - 2 + \frac{B\sqrt{-\log q}}{\sqrt{A}} < 0. \end{aligned}$$

Thus $h(x)$ is decreasing for $x \geq r$. Split J_2 into integrals over $[0, [r]]$ and $[[r], \infty)$. The second integral is bounded by the same arguments as above. The first integral can be estimated trivially by

$$2A \left| \int_0^{[r]} \left(x - [x] - \frac{1}{2}\right) q^{Ax^2+Bx} x dx \right| \leq A \int_0^{[r]} x dx \leq A[r]^2.$$

Therefore $\sqrt{-\log q}R_f^+ \rightarrow 0$ for $q \rightarrow 1$. Analogously we get $\sqrt{-\log q}R_f^- \rightarrow 0$. In order to show that the term with

$$R_g^+ = \int_0^{\infty} \left(x - [x] - \frac{1}{2} \right) \left(q^{Ax^2+Bx} + xq^{Ax^2+Bx} \log q(2Ax+B) \right) dx$$

vanishes, it remains to consider the integral

$$J_3 := \int_0^{\infty} \left(x - [x] - \frac{1}{2} \right) q^{Ax^2+Bx} qx^2 dx.$$

Again we determine where $H(x) := x^2 q^{Ax^2+Bx}$ is decreasing. We have

$$H'(x) = q^{Ax^2+Bx} (2x + (2Ax^3 + Bx^2) \log q)$$

and therefore $\lim_{x \rightarrow -\infty} H'(x) = +\infty$, $\lim_{x \rightarrow \infty} H'(x) = -\infty$ and $H'(0) = 0$. Since $H''(0) > 0$, there exists a single positive root s of $H'(x)$. Moreover $s \leq 1/\sqrt{-A \log q}$ since

$$\begin{aligned} H' \left(\frac{1}{\sqrt{-A \log q}} \right) &= \frac{2}{\sqrt{-A \log q}} + \log q \left(\frac{2a}{(-A \log q)^{3/2}} - \frac{B}{A \log q} \right) \\ &= \frac{2}{\sqrt{-A \log q}} - \frac{2}{\sqrt{-A \log q}} - \frac{B}{A} \leq 0. \end{aligned}$$

Thus $H(x)$ is decreasing for $x \geq s$. Split the integral into integrals over $[0, [s]]$, $[[s], [s]]$ and $[[s], \infty)$. The third integral is bounded as above. The second integral is trivially bounded by $\frac{1}{2}[s]^2$. And the first integral — the increasing part — we estimate with the same ideas as for the decreasing part:

$$\begin{aligned} 0 &\leq \sum_{i=0}^{[s]-1} \int_i^{i+1} \left(x - [x] - \frac{1}{2} \right) q^{Ax^2+Bx} qx^2 dx \\ &= \int_0^{[s]} \left(x - [x] - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 dx \\ &= \int_0^{\frac{1}{2}} \left(x - [x] - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 dx + \sum_{i=0}^{[s]-2} \int_{\frac{1}{2}+i}^{\frac{1}{2}+i+1} \left(x - [x] - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 dx \\ &\quad + \int_{[s]-\frac{1}{2}}^{[s]} \left(x - [x] - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 dx \\ &\leq 0 + \sum_{i=0}^{[s]-2} 0 + \frac{1}{4}[s]^2 \end{aligned}$$

Therefore $\sqrt{-\log q}R_g^+ \rightarrow 0$ for $q \rightarrow 1$. In a similar way we find $\sqrt{-\log q}R_g^- \rightarrow 0$.

After this analysis of the means, we turn our attention to the sequence of variances.

Lemma 5.5. *For $n \rightarrow \infty$ we have*

$$\sigma_n^2 = \phi(a, \alpha, q) - c(a, \alpha, q)^2 + o(1),$$

where

$$\phi(a, \alpha, q) := \frac{e_q(q)}{\gamma} \sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right).$$

Proof. By definition we have

$$\mathbb{E}(X_n^2) = \frac{\sum_{x=0}^n x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}{\sum_{x=0}^n \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}.$$

Now we proceed as in the proof of Lemma 5.2 and study the numerator \tilde{N} after expansion by $q^{\frac{f(n)^2}{4\alpha}}$.

$$\tilde{N} = \sum_{x=0}^n x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}};$$

we split the sum and reverse the order of summation resp. shift the summation index and get

$$\begin{aligned}
&= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} \left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - x \right)^2 \left[\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - x \right]_q q^{\alpha(a+x)^2} + \\
&\quad + \sum_{x=0}^{n - \lfloor \frac{f(n)}{2\alpha} \rfloor - 1} \left(x + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + 1 \right)^2 \left[x + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + 1 \right]_q q^{\alpha(a-(x+1))^2},
\end{aligned}$$

which can be written as

$$\begin{aligned}
&= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \Sigma_1 - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1(x) + \Sigma_1(x^2) + \Sigma_2(x^2) + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \Sigma_2 + \Sigma_2 \\
&\quad + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2(x) + 2\Sigma_2(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2.
\end{aligned}$$

Using similar arguments as above yields

$$\begin{aligned}
&= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \gamma + e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) \right. \\
&\quad \left. + \Sigma_1^\infty(x^2) + \Sigma_2^\infty(x^2) + \Sigma_2^\infty + 2\Sigma_2^\infty(x) \right) + o(1).
\end{aligned}$$

Thus $E(X_n^2)$ equals

$$\begin{aligned}
&\frac{1}{\gamma} \left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \gamma + e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) \right) \right) \\
&\quad + \phi(a, \alpha, q).
\end{aligned}$$

Since

$$\begin{aligned}
\mu_n^2 &= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 - \frac{e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty \right)}{\gamma} \\
&\quad + c(a, \alpha, q)^2 + o(1),
\end{aligned}$$

we obtain

$$\sigma_n^2 = \phi(a, \alpha, q) - c(a, \alpha, q)^2 + o(1). \quad \square$$

Lemma 5.6.

$$\phi(a, \alpha, q) > c(a, \alpha, q)^2.$$

Proof. We have to show that

$$\frac{\sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}} > \left(\frac{\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}} \right)^2.$$

A sufficient condition for this is

$$\frac{\sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}} > \left(\frac{\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}} \right)^2.$$

We show that

$$\begin{aligned}
&\left(\sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right) \right) \left(\sum_{x=1}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2} \right) \geq \\
&\quad \geq \left(\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right) \right)^2.
\end{aligned}$$

Expanding gives

$$\begin{aligned}
 & \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\
 & \quad + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} \\
 & \geq \sum_{x,y=1}^{\infty} x q^{\alpha(x+a)^2} y q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x q^{\alpha(x-a)^2} y q^{\alpha(y-a)^2} \\
 & \quad + 2 \sum_{x,y=1}^{\infty} x q^{\alpha(x+a)^2} y q^{\alpha(y-a)^2}.
 \end{aligned}$$

Now we consider the pairs (x, y) and (y, x) again and obtain

$$\begin{aligned}
 & x^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + y^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + x^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\
 & \quad + y^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + x^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + y^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\
 & \quad + x^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} + y^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} \\
 & \geq 2xy q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + 2xy q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} + 2xy q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\
 & \quad + 2xy q^{\alpha(x-a)^2} q^{\alpha(y+a)^2}.
 \end{aligned}$$

This is true since $x^2 + y^2 \geq 2xy$. \square

Now we are able to establish the next convergence result. For this purpose recall that $c(a, \alpha, q)$ and $\phi(a, \alpha, q)$ depend on the fractional part of $\frac{f(n)}{2\alpha}$. Since convergent variances and convergent fractional parts of means are required for convergence to a discrete distribution, we will choose a subsequence (n_k) of (n) such that $\{\frac{f(n)}{2\alpha}\}$ remains constant.

Theorem 5.7. *Let (n_k) be an increasing sequence of natural numbers and $X_{n_k} \sim \mathcal{B}(\alpha, q^{-f(n_k)}, n_k, q)$ such that $\{\frac{f(n)}{2\alpha}\} = a$ a constant. Recall that we assume $f(n) \rightarrow \infty$, $2\alpha n - f(n) \rightarrow \infty$ and $\alpha > 0$. Then $(X_{n_k} - \mu_{n_k})/\sigma_{n_k}$ converges for $k \rightarrow \infty$ to a normalised discrete normal distribution, i.e., they converge to a limit X with*

$$\mathbb{P}\left(X = \frac{1}{\sigma} (x - c(a, \alpha, q))\right) = \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2}},$$

where $\sigma = \lim_{k \rightarrow \infty} \sigma_{n_k}$ and $c(a, \alpha, q)$ is defined as in Lemma 5.2..

Proof. For simplicity we write in the following n instead of n_k . First we note that Lemmas 5.5 and 5.6 imply that the sequence of variances (σ_n^2) converges since $\{\frac{f(n)}{2\alpha}\}$ is constant by assumption. We define

$$H(\mu_n) := \begin{cases} \lfloor \mu_n \rfloor & \text{if } a > 0 \\ \lfloor \mu_n \rfloor & \text{if } a = 0, n \geq \frac{f(n)}{4\alpha} \\ \lceil \mu_n \rceil & \text{if } a = 0, n < \frac{f(n)}{4\alpha} \end{cases}.$$

Since $H(\mu_n) = \frac{f(n)}{2\alpha} - a$, we have

$$\begin{aligned}
 \mathbb{P}(X_n = H(\mu_n) + x) &= \left[\frac{f(n)}{2\alpha} - a + x \right]_q \frac{q^{\left(\frac{f(n)}{2\alpha} - a + x\right)^2 - f(n)\left(\frac{f(n)}{2\alpha} - a + x\right)}}{\sum_{y=0}^n \left[\frac{n}{y} \right]_q q^{\alpha y^2 - f(n)y}} \\
 &= \left[\frac{f(n)}{2\alpha} - a + x \right]_q \frac{q^{-\frac{f(n)^2}{4\alpha} + \alpha(x-a)^2}}{\sum_{y=0}^n \left[\frac{n}{y} \right]_q q^{\alpha y^2 - f(n)y}} \\
 &\rightarrow e_q(q) \frac{q^{\alpha(x-a)^2}}{e_q(q) \sum_{x=0}^{\infty} (q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2})} \\
 &= \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(a+x)^2}} \\
 &= \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2}}.
 \end{aligned}$$

By normalising we get the theorem. \square

For $\alpha = \frac{1}{2}$ this theorem reduces to the convergence property of Kemp's binomial distribution established in [6].

Using Jacobi's Triple Product ([5]) we can rewrite the infinite sum as

$$\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2} = q^{\alpha a^2} (q^{2\alpha}, q)_{\infty} (-q^{\alpha-2\alpha a}, q)_{\infty} (-q^{\alpha+2\alpha a}, q)_{\infty}.$$

In the limit $q \rightarrow 1$ these discrete normal distributions converge to the standard normal distribution, see [14].

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(Martin Zeiner) VIENNA UNIVERSITY OF TECHNOLOGY, TREITLSTRASSE 3/II, 1040 WIEN, AUSTRIA
E-mail address: `zeiner at math.tugraz.at`